LONGITUDINAL SHEAR CRACK WITH AN INFINITELY NARROW PLASTIC ZONE

(TRESHCHINA PRODOL'NOGO SDVIGA S BESKONECHNO UZKOI PLASTICHESLOI ZONOI)

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B. V. KOSTROV and L. V. NIKITIN
(Moscow)

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Taking account of plasticity effects in problems on crack equilibrium in solids is an important but very complex problem. Proceeding from experimental observations, Dugdale [1] proposed consideration of the plastic zone near the crack tip to be infinitely narrow. The solution he obtained on the basis of this assumption disclosed good enough agreement with experiment in the plane stress state [2 and 3]. In this connection, the Dugdale hypothesis received a rather wide acceptance [4 to 9]. For example, on the basis of the Dugdale hypothesis, Field [6] considered the problem of a longitudinal shear crack by assuming, as did Dugdale, that one of the stress tensor components on the plastic zone boundary is equal to the yield point. However, the latter condition is not a physical one in the longitudinal shear case, and results in exceeding the yield point in the elastic domain. A solution is constructed below for a longitudinal shear crack, which is also based on the Dugdale hypothesis, however, compliance with the Mises plasticity condition on the plastic zone boundary is required, and this permits elimination of the mentioned inadequacy.

Let us consider a loading-free crack of length $2x_0$ in an unbounded elastic medium in an anti-plane state of strain subject to the stress $\tau_{yz} = \tau_\infty$ at infinity. Let the crack be located at $y = 0$, $|x| < x_0$. It may be shown that the complex stress function $\tau = \tau_y + i\tau_x$ where $\tau_x = \tau_{xz}$, $\tau_y = \tau_{yz}$ will be an analytic function of the complex variable $\zeta = x + iy$, which is regular outside the real axis. By virtue of the symmetry of the problem, only the right half-plane $\zeta(\mathrm{Re}\,\zeta \geq 0)$ need be considered. It may be shown that the inverse function $\zeta = \zeta(\tau)$ is hence a single-valued analytic function of τ.

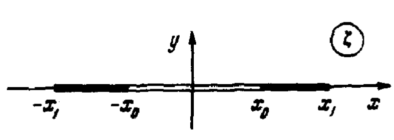

Fig. 1

Following the idea of Dugdale, let us assume that the plastic zones at the ends of the cracks (Fig. 1) are segments of the x-axis, so that $x_0 < |x| < x_1$, $y = 0$. Let us require compliance with the plasticity condition

$$|\tau| \equiv \sqrt{\tau_x^2 + \tau_y^2} = k \qquad (1)$$

at the plastic zone boundaries, where k is the yield point.

The condition $|\tau| \leq k$ should be satisfied in the whole elastic domain. Condition (1) differs from the condition $\tau_y = k$ used by Field [4], which permits avoiding the aforementioned difficulty. The function $\zeta = \zeta(\tau)$ maps the semicircle $\mathrm{Re}\,\tau \geq 0$, $|\tau| \leq k$ with the slit (Fig. 2) along the real axis segment $0 \leq \mathrm{Re}\,\tau \leq \tau_\infty$ into the right ζ half-plane with a slit along the segment $0 \leq x \leq x_1$.

It is easy to see that the following boundary conditions hold :

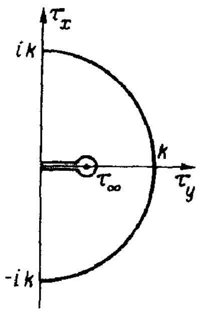

Fig. 2

$$\operatorname{Im}\zeta = 0 \quad \text{for} \quad \begin{cases} \operatorname{Re}\tau = 0, & -k < \operatorname{Im}\tau < k \\ |\tau| = k, & -\tfrac{1}{2}\pi < \arg\tau < \tfrac{1}{2}\pi \\ \operatorname{Im}\tau = 0, & \tau_\infty < \operatorname{Re}\tau < k \end{cases} \tag{2}$$

$$\operatorname{Re}\zeta = 0 \ \text{ for } \ \operatorname{Im}\tau = 0,\ 0 \leqslant \operatorname{Re}\tau < \tau_\infty$$

It hence follows, in particular, that

$$\zeta = O\left(\frac{1}{\sqrt{\tau_\infty - \tau}}\right) \qquad \text{for } \tau \to \tau_\infty$$

The conditions (2) define the Hilbert problem in the function $\zeta(\tau)$. The solution of this problem is

$$\zeta = \frac{x_0(\tau_\infty^2 + k^2)\tau}{\sqrt{(k^4 - \tau^2\tau_\infty^2)(\tau^2 - \tau_\infty^2)}} \tag{3}$$

Putting $\tau = k$ here, we find the position of the edge and the length of the plastic zone

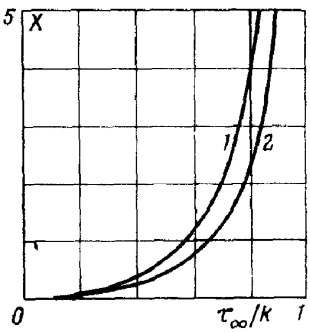

Fig. 3

$$x_1 = x_0\frac{k^2 + \tau_\infty^2}{k^2 - \tau_\infty^2}, \qquad x_1 - x_0 = x_0\frac{2\tau_\infty^2}{k^2 - \tau_\infty^2} \tag{4}$$

The Field solution [6] in the notation used here yields

$$x_1 - x_0 = x_0\left(\sec\frac{\pi\tau_\infty}{2k} - 1\right) \tag{5}$$

Shown in Fig. 3 are curves 1 and 2 computed by means of Formulas (4) and (5), respectively, where $X = (x_1 - x_0)/x_0$. From a comparison it is seen that corresponding dimensions of the plastic zones may differ noticeably.

Now, let us calculate the displacement u of the medium. The displacement u may be represented as the real part of an analytic function $U(\zeta)$ connected with the function τ by the obvious relationship $\tau = \mu i U'(\zeta)$. There hence follows from the solution (3)

$$dU = \frac{ix_0\tau_\infty^2(k^2 + \tau_\infty^2)(k^4 - \tau^4)\tau}{\mu(k^4 - \tau_\infty^2\tau^2)^{3/2}(\tau^2 - \tau_\infty^2)^{3/2}}\,d\tau$$

Integrating this expression we find (6)

$$u = \frac{x_0(k^2+\tau_\infty^2)}{\mu}\left[\frac{1}{2\tau_\infty}\ln\left|\frac{\sqrt{k^4 - \tau_\infty^2\tau^2} + i\tau_\infty\sqrt{\tau^2 - \tau_\infty^2}}{\sqrt{k^4 - \tau_\infty^2\tau^2} - i\tau_\infty\sqrt{\tau^2 - \tau_\infty^2}}\right| + \operatorname{Im}\frac{\tau^2}{\sqrt{(k^4 - \tau_\infty^2\tau^2)(\tau^2 - \tau_\infty^2)}}\right]$$

The stresses and displacements along the boundary of the plastic zone may be obtained from (3) and (6) by substituting $|\tau| = k$, which yields

$$\tau_y = \frac{(k^2 + \tau_\infty^2)\sqrt{x^2 - x_0^2}}{2\tau_\infty x}, \qquad \tau_x = \frac{\mp(k^2 - \tau_\infty^2)\sqrt{x_1^2 - x^2}}{2\tau_\infty x}$$

$$u = -\frac{x_0(k^2 + \tau_\infty^2)}{2\mu\tau_\infty}\left(\frac{1}{2}\ln\frac{x_1 - \sqrt{x_1^2 - x^2}}{x_1 + \sqrt{x_1^2 - x^2}} - \frac{\sqrt{x_1^2 - x^2}}{x_1}\right) \qquad (y = \pm 0) \tag{7}$$

The discontinuity in the displacement at the terminus of the crack $x = x_0$ will be

$$u^+ - u^-|_{x=x_0} = -\frac{x_0(k^2 + \tau_\infty^2)}{\mu\tau_\infty}\left(\ln\frac{k - \tau_\infty}{k + \tau_\infty} + \frac{2k\tau_\infty}{k^2 + \tau_\infty^2}\right) \tag{8}$$

Fig. 4 presents the dependence of the displacement at the root of the crack on τ_∞ / k (curve 1) together with the corresponding dependence obtained on the basis of the Field's paper [6] (curve 2). Displacements along the free portion of the crack and the plastic zone are shown in Fig. 5, where $u^* = \mu u / k x_0$.

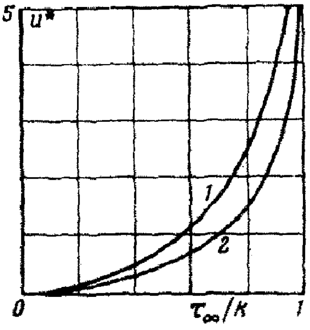

Fig. 4

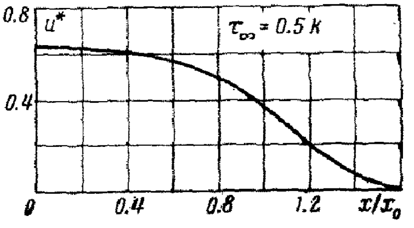

Fig. 5

Using the principle of the maximum of the modulus for analytic functions, it may be shown that the plasticity condition is not violated for the constructed solution.

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